3. de Arantes e Oliveira, E. R., Theoretical foundations of the finite element method. Int. J. Solids and Structures, Pergamon Press, Vol. 4, №10, 1968.
4. Zenisek, A and Zlamal, M., Convergence of a finite element procedure for solving boundary value problems of the fourth order. Int. J. for Numerical Methods in Engng. Vol. 2, №3, 1970.
5. Leckie, F. A. and Lindberg, G. M., The effect of lumped parameters on beam frequencies. Aeronaut. Quarterly, Vol. 14, №3, 1963.
6. Kandidov, V. P. and Kim, L. P., Discrete model for the investigation of beam vibrations. Vestnik Moskov. Univ., Fiz., Astron., №5, 1968.
7. Olson, M. D. and Lindberg, G. M., Convergence studies of eigenvalue solutions using two finite plate bending elements. Int. J. Mech. Sci., Vol. 12, №1, 1970.
8. Mei Chuh, Free vibration of circular membranes under arbitrary tension by the finite-element method. J. Acoust. Soc. Amer., Vol. 46, №3, 1969.
9. Durwasula, S., Natural frequencies and modes of skew membranes. J. Acoust. Soc. Amer., Vol. 44, №6, 1968.

Translated by M. D. F.

UDC 539.3 : 534.1

TORSIONAL VIBRATIONS OF A VISCOELASTIC HALF-SPACE

PMM Vol. 36, №3, 1972, pp. 565-568

V.P. TEN
(Moscow)

(Received October 18, 1971)

A circular rigid stamp is in contact with the surface of a viscoelastic half-space. The stamp performs forced harmonic oscillations around the axis of symmetry. The surface is stress- free everywhere outside the domain of contact.

Approximate expressions are found for the displacement, the stress under the stamp, the moment of the reactive forces acting on the stamp under the assumption of stationarity of the oscillations and their disappearance at infinity.

Sagoci [1] obtained the solution of an analogous problem for an elastic half-space.

1. Let us introduce a cylindrical r, φ, z coordinate system with origin at the center of the contact domain. By analogy with the elastic problem, only the following quantities are not trivial: u_φ, $\tau_{z\varphi}$, $\tau_{r\varphi}$ the angular displacement and the tangential stresses. The motion of the medium is described by the equation for the elastic displacement with two boundary conditions

$$\left[\mu - \int_{-\infty}^{t} K(t-\tau)\,d\tau\right]\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} + \frac{\partial^2 u_\varphi}{\partial z^2}\right) = \rho\frac{\partial^2 u_\varphi}{\partial t^2}$$

$$u_\varphi = r\,\Phi \exp(i\omega t) \quad \text{for } z = 0,\ r < R$$

$$\tau_{z\varphi} = \left[\mu - \int_{-\infty}^{t} K(t-\tau)\,d\tau\right]\frac{\partial u_\varphi}{\partial z} = 0 \quad \text{for } z = 0,\ r > R$$

where ρ is the density of the medium, R is the radius of the stamp, μ is the instantaneous shear modulus, K is the hereditary kernel, ω is the frequency, and Φ is the amplitude of the oscillations.

In the case of steady oscillations, all the desired quantities can be represented as the product of the complex amplitude by an exponential. The effect of the Volterra operator on such functions is equivalent to multiplication by some complex number

$$\mu\psi(r, z)e^{i\omega t} - \int_{-\infty}^{t} K(t-\tau)\psi(r, z)e^{i\omega\tau}d\tau = \mu_*\psi(r, z)e^{i\omega t}$$

$$\mu_* = \mu_1 + i\mu_2 = \left[\mu - \int_0^\infty K(x)\cos(\omega x)\,dx\right] + i\int_0^\infty K(x)\sin(\omega x)\,dx$$

Here μ_* denotes the complex shear modulus. Therefore, the amplitude of the displacements must satisfy an equation with the boundary conditions

$$\mu_*\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) = -\rho\omega^2 u \tag{1.1}$$

$$u = r\,\Phi \quad \text{for } z = 0,\ r < R;\ \partial u / \partial z = 0,\ \text{ for } z = 0,\ r > R$$

The substitution $w = u\exp(i\varphi)$ transforms (1.1) into a Helmholtz equation for w with the complex parameter $k^2 = \rho\omega^2 / \mu_*$. The boundary conditions become

$$w = r\Phi\exp(i\varphi) \ \text{ for } z = 0,\ r < R;\ \partial w / \partial z = 0 \ \text{ for } z = 0,\ r > R$$

2. Let us introduce spheroidal ξ, η, φ coordinates related to the cylindrical coordinates by means of the following expressions:

$$r = R\,[(1+\xi^2)(1-\eta^2)]^{1/2},\quad z = R\xi\eta,\quad \varphi = \varphi\ (0 \leqslant \xi < \infty,\ |\eta| \leqslant 1)$$

The domain $\xi = 0$ corresponds to the interior of the circle $z = 0, r < R$, and the domain $\eta = 0$ to its exterior. The boundary conditions go over into $w(0, \eta) = R\Phi(1 - \eta^2)^{1/2}e^{i\varphi}$, $w_\eta'(\xi, 0) = 0$. Assuming $w = U(\xi)\,V(\eta)\exp(im\varphi)$, we obtain an equation for V

$$[(1-\eta^2)V'(\eta)]' + [\lambda + \theta(1-\eta^2) - m^2/(1-\eta^2)]\,V(\eta) = 0 \tag{2.1}$$

$$(\theta = -\rho\omega^2 R^2 / \mu_*)$$

Here λ is the eigenvalue and θ is a given dimensionless parameter.

An analogous equation for U is reduced to (2.1) by the substitution $\xi = i\eta$, hence it is sufficient to study (2.1) for any complex η. There results from the boundary conditions that $m = 1$, and $V'(0) = 0$. Therefore, a Sturm-Liouville problem originates for (2.1). Equation (2.1) has been studied in detail in [2] by perturbation theory methods. The eigenfunctions are sought as series of Bessel functions for $|\eta| > 1$. Within the circle $|\eta| < 1$ the expansions are constructed with Legendre functions.

In the case of an elastic medium (the Sagoci problem) θ is real. For a viscoelastic medium θ takes on complex values. Let us assume that $\operatorname{Im}\sqrt{\theta} > 0$. Then as $\mu_2 \to 0$ the solution for the viscoelastic medium goes over into the Sagoci solution. We retain the notation from [2] with the sole difference that here θ corresponds to 4θ in [2].

The condition for the disappearance of the solution at infinity together with the conditions $\operatorname{Im}\sqrt{\theta} > 0$ and $V(0) = 0$ extracts the necessary set of spheroidal wave functions (see [2]) $Ps_{1+2n}^1(\eta, \theta)$ and $S_{1+2n}^{1(4)}(-i\xi, \theta)$. The series

$$w = \sum_{n=0}^{\infty} A_n S^{1(4)}_{1+2n}(-i\xi, \theta)\, \mathrm{P}s^1_{1+2n}(\eta, \theta)\, e^{i\varphi}$$

satisfies the Helmholtz equation and the boundary condition outside the domain of contact. To determine the coefficients A_n we use the condition on the displacement under the stamp

$$u(0, \eta) = R\Phi(1-\eta^2)^{1/2} = \sum_{n=0}^{\infty} A_n S^{1(4)}_{1+2n}(-i0, \theta)\, \mathrm{P}s^1_{1+2n}(\eta, \theta)$$

Since the $\mathrm{P}s^1_{1+2n}$ are orthogonal, then

$$A_n = R\Phi \left[\int_{-1}^{1} (1-\eta^2)^{1/2}\, \overline{\mathrm{P}s^1_{1+2n}}\, d\eta\right] \Big/ \left[S^{1(4)}_{1+2n}(-i0, \theta) \int_{-1}^{1} |\mathrm{P}s^1_{1+2n}|^2\, d\eta\right]$$

Now, if we use the expansion (see [2])

$$\mathrm{P}s^1_{1+2n}(\eta, \theta) = \sum_{k=0}^{\infty} (-1)^{n+k} a^1_{1+2n,\, k-n}(\theta)\, \mathrm{P}^1_{1+2k}(\eta)$$

and the fact that $(1-\eta^2)^{1/2} = \mathrm{P}_1{}^1(\eta)$, then both integrals can be expressed in terms of the coefficients $a^1_{m,n}$. Therefore

$$A_n = {}^4/_3\, R\Phi\, (-1)^n\, \bar{a}^1_{1+2n,\,-n}(\theta) \,/\, [S^{1(4)}_{1+2n}(-i0, \theta)\, \| \mathrm{P}s^1_{1+2n} \|^2]$$

Let us present the expressions for some quantities

$$\tau_{z\varphi}(0, \eta) = \frac{4}{3\eta}\, \mu_* \Phi e^{i\omega t} \sum_{n=0}^{\infty} \frac{(-1)^n \bar{a}^1_{1+2n,\,-n} S'^{1(4)}_{1+2n}(-i0, \theta)}{S^{1(4)}_{1+2n}(-i0, \theta)\, \| \mathrm{P}s^1_{1+2n} \|^2}\, \mathrm{P}s^1_{1+2n}(\eta, \theta)$$

$$M = 2\pi\mu_* R^3 \Phi e^{i\omega t} \int_0^1 \tau_{z\varphi}(0, \eta)(1-\eta^2)^{1/2}\, \eta\, d\eta =$$

$$= \frac{16}{9}\, \pi\mu_* R^3 \Phi e^{i\omega t} \sum_{n=0}^{\infty} \frac{|a^1_{1+2n,\,-n}|^2 S'^{1(4)}_{1+2n}(-i0, \theta)}{S^{1(4)}_{1+2n}(-i0, \theta)\, \| \mathrm{P}s^1_{1+2n} \|^2}$$

Evidently the amplitude of the stresses grows without limit upon approaching the edge of the stamp.

3. Existing tables of the coefficients of expansions of the spheroidal wave functions have been constructed only for real θ. To obtain approximate results in the complex domain, let us assume that the parameter θ is small in absolute value. By using the eigenvalue λ_n expansions presented in [3], the coefficients of the expansions of spheroidal functions as power series in θ can be determined to the accuracy of a constant complex factor. These coefficients are evidently also real for real θ. If it is required in addition that $\mathrm{Re}\, a^1_{m,0}(\theta) > 0$ and $a^1_{m,0}(0) = 1$, then they are constructed uniquely. Omitting detailed computations, let us present the principal terms of the following quantities.

$$u_\varphi = \frac{2(1-1.26\theta)(1-0.4\bar{\theta})}{\pi[1-0.4(\theta+\bar{\theta})]\,\Theta(\theta)}\, R\Phi e^{i\omega t} Q_1{}^1(-i\xi)\, \mathrm{P}_1{}^1(\eta) + \ldots \tag{3.1}$$

$$u_\varphi = -\frac{4\sqrt{\theta}(1-1.28\theta)(1-0.4\bar{\theta})\, r}{3\pi[1-0.4(\theta+\bar{\theta})]\,\Theta(\theta)(1+\xi^2)} \left(1 + \frac{1}{\sqrt{\theta}\,\xi}\right) e^{-\sqrt{\theta}\,\xi} \Phi e^{i\omega t} + \ldots \tag{3.2}$$

$$(\xi \gg 1)$$

$$\tau_{z\varphi}(0, \eta) = \frac{4(1-0.64\theta)}{\pi\Theta(\theta)\eta}\mu_* \Phi e^{i\omega t} P_1^{\ 1}(\eta) + \ldots \tag{3.3}$$

$$M = \frac{16(1-0.64\,\theta)}{3\Theta(\theta)}\mu_* R^3 \Phi e^{i\omega t} + \ldots \tag{3.4}$$

$$[\Theta(\theta) = 1 - 0.84\theta - 0.142\theta^{3/2}(1-0.48\theta)]$$

For real θ we can compare (3.1) - (3.4) with the Sagoci results. Assuming $\theta = -0.16$ say, we obtain from (3.4)(in the Sagoci notation) $-\mathrm{tg}\Delta = 0.00858$, $-\pi g_1 = 0.606$, $\pi g_2 = 0.0052$, which agrees with the appropriate points of the graphs presented in [1].

It follows from (3.1) and (3.3) that the displacements of an elastic and viscoelastic medium agree for $\omega = 0$, and the stresses are proportional. This is because only steady states are considered, which set in after a long time has elapsed following application of the load.

The solution for a concentrated moment can be obtained from (3.2) and (3.4). Let R tend to zero, and ξ to infinity so that $R\xi$ tends to a quantity on the order of the distance from the point of observation to the point of application of the moment. Hence, let us magnify the stress under the stamp without limit so that the moment remains constant in amplitude. Eliminating the quantity $R^3\Phi$, from (3.2) and (3.4), we obtain

$$u_\varphi = -\frac{Mr}{4\pi\mu_* R^2}\left(\frac{1}{R} + ik\right)e^{i(\omega t - kR)}$$

$$(R = \sqrt{r^2+z^2}, \quad k = k_1 - ik_2)$$

Here k is understood to be the branch of $\sqrt{k^2}$ with positive real part. Since $\mu_2 > 0$, then also $k_2 > 0$, hence the displacement damps out rapidly with distance from the point of application of the concentrated moment.

BIBLIOGRAPHY

1. Sagoci, H., Forced torsional oscillations of an elastic half-space. II. J. Appl. Phys., Vol. 15, 1944.
2. Bateman, H. and Erdélyi, A., Higher Transcendental Functions. Elliptic and Automorphic Functions. Lamé and Mathieu Functions. Moscow, "Nauka", 1967.
3. Meixner, J. and Schäfke, F. W., Mathieusche Funktionen und Sphäroidfunktionen. Berlin, Springer, 1954.

Translated by M. D. F.